Degeneracy and generalized solutions of optimal control problems*)

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Abstract

Degenerate optimal control problems are often met in applications and bring serious difficulties for general methods. They were studied systematically beginning from the 1960-s [1-4], and defined (generalizing the classical notion of degeneracy) as those which have some latent passive differential constraints. We propose special methods based, in essence, on detection and elimination of passive constraints. The core of this theory is a certain transformation of systems

$$\dot{x} = f(t, x, u), \quad t \in \mathbf{T} = [t_I, t_F],\ x \in R^n,\ u \in \mathbf{U}(t, x) \subset R^p \tag{1}$$

with unbounded velocity set $\mathbf{V}(t, x) = f(t, x, \mathbf{U}(t, x))$. Namely, we introduce an auxiliary system which describes the behavior of the original system for large velocities (called *limit system*):

$$\frac{dx}{d\tau} = l \in \mathbf{K}(t, x). \tag{2}$$

Here t is a parameter and $\mathbf{K}$ is the union of all limits $l = \lim v_q |v_q|^{-1}$ as $|v_q| \to \infty,\ \{v_q\} \subset \mathbf{V}$.

Let $y = \eta(t, x), \quad y \in R^m,\ 0 < m < n$, be the greatest full controllability multitude of (3). We construct the system (called *derived system*):

$$\dot{y} = \eta_x f(t, x, u) + \eta_t, \quad u \in \mathbf{U}(t, x), \quad y = \eta(t, x). \tag{3}$$

The set $\mathbf{E}$ of piecewise continuous functions $\hat{x}(t)$ satisfying (3), is an extension of the set $\mathbf{D}$ of piecewise smooth $x(t)$ satisfying (1). The following statement is true (under some natural assumptions).

Theorem 1. For any $\hat{x}(t) \in \mathbf{E}$ there exists a sequence $\{x_s(t)\} \subset \mathbf{D}$ converging to $\hat{x}(t)$ in measure on a prescribed bounded interval $\mathbf{T}$ with $x_s(t_\alpha) \to \hat{x}(t_\alpha)$ for any prescribed finite set of values $\{t_\alpha\} \subset \mathbf{T}$.

A constructive proof of the theorem is given in [4].

Any solution of the derived system $x(t)$ is viewed as a generalized solution of the original system called the *impulse mode*. In general it will be the *impulse sliding mode*. However for the class of the systems with unbounded linear control, typical generalized solutions are piecewise continuous with $x(t)$ containing a finite number of impulses on any bounded time interval. Each continuous section of $x(t)$ satisfying the original systems is called a *turnpike*. Stated differently, this solution can be treated as a motion along turnpikes with fast (instantaneous in the limit terms) transitions from one turnpike to another.

Of course, the first representatives of degenerate problems are free-end optimal control problems, stated for the above systems, with an unbounded velocity set. We eliminate the

passive constraints by replacing the original differential constraints by the derived system and obtain an equivalent *derived problem* of reduced order. A series of important theoretical results have been obtained in this way, including concrete nonlocal optimality conditions and improved algorithms for the search of singular and generalized solutions.

The turnpike approach is also applicable, both in exact or approximate form, to the degenerate problems with well bounded control. It is closely connected with a special method of constructing Krotov's function called *method of multiple maxima* [2, 3].

Analogues for time-discrete systems have been also developed. We shall also address applications to periodical systems on unbounded time intervals and give a brief survey of other efficient applications to various classes of practical problems.

Thus, the "evil" of degeneracy for the general methods turns into a "blessing" for special methods based on the possibility to lower the order of the problem with its simultaneous regularization.

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References

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