Second-order conditions in optimal control

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Abstract

In the lecture, we present a review of the author's results on second order conditions in optimal control, obtained during last 30 years. We formulate these conditions for three classes of optimal control problems. The first class is presented by the following *problem* A, considered on a variable time interval $[t_0, t_1]$, with initial-final time-state equality and inequality constraints and mixed state-control constraints such that the gradients w.r.t. control of active constraints are *linearly independent*:

$$\mathcal{J} = J(t_0, x(t_0), t_1, x(t_1)) \longrightarrow \min, \quad F(t_0, x(t_0), t_1, x(t_1)) \le 0, \quad K(t_0, x(t_0), t_1, x(t_1)) = 0,$$
$$\dot{x}(t) = f(t, x(t), u(t)), \quad g(t, x(t), u(t)) = 0, \quad \varphi(t, x(t), u(t)) \le 0 \ \text{ for a.a. } t \in (t_0, t_1),$$
$$(t_0, x(t_0), t_1, x(t_1)) \in \mathcal{P}, \ (t, x(t), u(t)) \in \mathcal{Q} \ \text{ for a.a. } t \in (t_0, t_1).$$

Here $\mathcal{P}$ and $\mathcal{Q}$ are open sets; the functions J, F, and K are twice continuously differentiable on $\mathcal{P}$; the functions f, g, and φ are twice continuously differentiable on $\mathcal{Q}$; the gradients with respect to control $g_{iu}(t, x, u)$, $i = 1, \dots, d(g)$ and $\varphi_{ju}(t, x, u)$, $j \in I_\varphi(t, x, u)$ are jointly linearly independent at each point $(t, x, u) \in \mathcal{Q}$ such that $g(t, x, u) = 0$, $\varphi(t, x, u) \le 0$, where $I_\varphi(t, x, u) = \{j \in \{1, \dots, d(\varphi)\} \mid \varphi_j(t, x, u) = 0\}$ is the set of active indices; $d(\varphi)$ denotes the dimension of the vector φ. Minimum is sought among trajectories $\mathcal{T} = (x(t), u(t) \mid t \in [t_0, t_1])$ such that the state $x(t)$ is absolutely continuous and the control $u(t)$ is measurable and essentially bounded on $[t_0, t_1]$. We formulate no-gap second order conditions of weak, strong and Pontryagin minima in this problem in the cases of continuous [1] and discontinuous [2] optimal control. Sufficient conditions guarantee the growth of the cost function of the definite "order", which is quadratic in the case of continuous optimal control and has more complicated, non-homogeneous structure in the case of discontinuous optimal control. We also discuss recent results on quadratic growth of the cost function in a special case of problem A

under assumption of *positive linear independence* of gradients w.r.t. control, obtained jointly with F. Bonnans.

The second class is presented by *problem* B considered on a non-fixed time interval $[t_0,\, t_1]$ and such that the control enters linearly in system dynamics and the control constraint is given by a convex polyhedron U. The problem has the form

$$\begin{aligned}
&\mathcal{J} = J(t_0, x(t_0), t_1, x(t_1)) \longrightarrow \min, \quad F(t_0, x(t_0), t_1, x(t_1)) \le 0, \quad K(t_0, x(t_0), t_1, x(t_1)) = 0,\\
&\dot{x}(t) = f(t, x(t), u(t)), \ \text{ for a.a. } t \in (t_0, t_1), \ \text{ where } f(t, x, u) = a(t, x) + B(t, x)u\\
&u(t) \in U \ \text{ for a.a. } t \in (t_0, t_1),\\
&(t_0, x(t_0), t_1, x(t_1)) \in \mathcal{P}, \ (t, x(t), u(t)) \in \mathcal{Q} \ \text{ for a.a. } t \in (t_0, t_1).
\end{aligned}$$

We formulate both necessary and sufficient second order conditions [3] of optimality of bang-bang control in problem B. The sufficient conditions in this problem are not a special case of those in problem A because (due to the linearity of the problem B w.r.t. control) the strengthened Legendre condition (which is a part of the sufficient conditions in problem A) is never fulfilled in problem B. Nevertheless a special method due to A.A. Milyutin [3] allows to derive sufficient conditions in problem B on the base of sufficient conditions in problem A. We discuss the relationship of our sufficient conditions in problem B with sufficient conditions of A.A. Agrachev, G. Stefany and P.L. Zezza. This relationship was clarified in our joint works with H. Maurer [5].

Finally the third class is presented by *problem* C, linear in a part of controls:

$$\begin{aligned}
&\mathcal{J} = J(t_0, x(t_0), t_1, x(t_1)) \longrightarrow \min, \quad F(t_0, x(t_0), t_1, x(t_1)) \le 0, \quad K(t_0, x(t_0), t_1, x(t_1)) = 0,\\
&\dot{x}(t) = f(t, x(t), u(t), v(t)) \text{ for a.a. } t \in (t_0, t_1), \ \text{ where } f(t, x, u, v) = a(t, x, v) + B(t, x, v)u,\\
&u(t) \in U \ \text{ for a.a. } t \in (t_0, t_1),\\
&(t_0, x(t_0), t_1, x(t_1)) \in \mathcal{P}, \ (t, x(t), u(t)) \in \mathcal{Q} \ \text{ for a.a. } t \in (t_0, t_1).
\end{aligned}$$

We formulate second order optimality conditions for the trajectory $\mathcal{T} = (x(t), u(t), v(t) \mid t \in [t_0, t_1])$ such that the control $u(t)$ is bang-bang and the control $v(t)$ is continuous on $[t_0,\, t_1]$. The joint works of the author with F. Lempio [4] and H. Maurer [5] allowed to extend the Riccati approach (used for analysis of second order sufficient conditions) to the case of discontinuous controls in the problems A, B, and C.

Key Words: maximum principle, broken extremal, strong minimum, jump of control, quadratic form, critical cone, sufficient optimality condition, Riccati equation, perfect square

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