Curvature terms in the use of coordinates as controls in Classical Mechanics

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Abstract

Let us consider a $(N+M)$-dimensional mechanical system locally parameterized by state coordinates $(q^1,\dots,q^N,u^1,\dots,u^M)$, and let us regard some of the latter, say $(u^1,\dots,u^M)$, as *controls*. This reduces the dimension of the system to N. For instance, two rigidly connected material points $\{P_1,P_2\}\in\mathbf{R}^2$ give rise to a three-dimensional system. Yet, if we prescribe the motion of P_1, $\{P_1,P_2\}$ becomes a one-dimensional system.

More generally, if $\mathcal{Q}$ and $\mathcal{U}$ are differential manifolds and the state-space coincides with the product $\mathcal{Q}\times\mathcal{U}$, one can regard $\mathcal{Q}$ as the actual (reduced) state space by considering $\mathcal{U}$ as a set of controls. Let us remark that the predetermination of the motion on the factor space $\mathcal{U}$ is nothing but an instance of what in Classical Mechanics is called *the imposition of a moving holonomic constraint*. The resulting equations are naturally set on T^*Q and have controls taking values in $T\mathcal{U}$. This means that the dynamics depends on the controls u^α and on their derivatives $\dot u^\alpha$ as well.

Notice that the parameters $\dot u^\alpha$ —to be considered as supplementary controls— may well be unbounded. In addition, the dynamics depends quadratically on them. Therefore, an important related issue consists in the characterization of the closure of the solutions' set for control systems of this kind.

At the same time, some remarkable geometric aspects have to be considered. In particular, these involve the kinetic metric and its relations with the foliation $\{\mathcal{Q}\times\{\mathbf{u}\}\mid \mathbf{u}\in\mathcal{U}\}$. In particular, the fact that a certain curvature term connected with this foliation is non-vanishing turns out to be a necessary condition for the stabilization of the system by rapid small oscillations. Such a condition becomes also sufficient provided some further sign assumptions are fulfilled. Let us remark that this includes the well-known stability of the *inverted pendulum* subject to rapid vertical oscillations of its pivot.

Key Words: Classical Mechanics, unbounded controls, Riemannian metrics